

Pushouts of categories, derived limits and colimits*

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Abstract

In a paper by Ford, it is claimed that to any pushout square of categories with all involved functors injective, there is associated an exact “Mayer–Vietoris” sequence of derived (co)limits. We provide a counter-example to this general statement. Further, we construct the Mayer–Vietoris sequence under some restrictions that cover the well-known case of a pushout square of group monomorphisms.

1. Introduction

In this paper, we are interested in derived functors of the colimit functor $\operatorname{colim}^{\mathcal{C}}: [\mathcal{C}, \mathbf{Ab}] \rightarrow \mathbf{Ab}$, defined on the category $[\mathcal{C}, \mathbf{Ab}]$ of functors $\mathcal{C} \rightarrow \mathbf{Ab}$; we denote them by $\operatorname{colim}_n^{\mathcal{C}} = L_n \operatorname{colim}^{\mathcal{C}}$.

We study the dependence of these derived colimits on the indexing category \mathcal{C} . More concretely, assume that \mathcal{C} is a pushout of two subcategories $\mathcal{C}_1, \mathcal{C}_2 \subseteq \mathcal{C}$ along their common subcategory \mathcal{C}_0 . It was claimed in [1] that in this situation, there exists, for $M: \mathcal{C} \rightarrow \mathbf{Ab}$, an exact *Mayer–Vietoris sequence*

$$\cdots \rightarrow \operatorname{colim}_{n+1}^{\mathcal{C}} M \xrightarrow{\partial} \operatorname{colim}_n^{\mathcal{C}_0} M \rightarrow \operatorname{colim}_n^{\mathcal{C}_1} M \oplus \operatorname{colim}_n^{\mathcal{C}_2} M \rightarrow \operatorname{colim}_n^{\mathcal{C}} M \xrightarrow{\partial} \operatorname{colim}_{n-1}^{\mathcal{C}_0} M \rightarrow \cdots,$$

where the derived colimits are taken of the restriction of M to the various subcategories of \mathcal{C} and where the maps are induced by the inclusions among the subcategories, with the exception of those denoted by ∂ . Another Mayer–Vietoris sequence was claimed for derived limits. The statements do not hold in this generality and we provide a simple counter-example in Section 3.

After that, we prove the following restricted version of the theorem.

Theorem 1. *Let*

$$\begin{array}{ccc} \mathcal{C}_0 & \xrightarrow{F_1} & \mathcal{C}_1 \\ F_2 \downarrow & & \downarrow I_1 \\ \mathcal{C}_2 & \xrightarrow{I_2} & \mathcal{C} \end{array} \quad (\text{Cat})$$

be a pushout square of categories with at least one of F_1, F_2 injective on objects and all I_1, I_2 and $I_1 F_1 = I_2 F_2$ local coverings. Then an exact Mayer–Vietoris sequence associated with this square exists.

Local coverings will be defined later in Section 5. They are generalizations of monomorphisms of groups. In particular, the above theorem generalizes the well-known special case of groups, see [2, 1B.12].

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2. Preliminaries on derived limits and colimits

Let \mathcal{C} be a small category. We denote by $\mathcal{C}\text{-Mod}$ the category of functors $\mathcal{C} \rightarrow \mathbf{Ab}$, also called *left \mathcal{C} -modules*. Dually, $\mathbf{Mod}\text{-}\mathcal{C}$ is the category of *right \mathcal{C} -modules*, i.e. functors $\mathcal{C}^{\text{op}} \rightarrow \mathbf{Ab}$. It is well-known that $\mathcal{C}\text{-Mod}$ has enough projectives and injectives and thus, one may define derived functors. In this paper, we are interested in the left derived functors $\text{colim}_n^{\mathcal{C}} = L_n \text{colim}^{\mathcal{C}}$ of the colimit functor $\text{colim}^{\mathcal{C}}: \mathcal{C}\text{-Mod} \rightarrow \mathbf{Ab}$ and right derived functors $\text{lim}_{\mathcal{C}^{\text{op}}}^n = R^n \text{lim}_{\mathcal{C}^{\text{op}}}$ of the limit functor $\text{lim}_{\mathcal{C}^{\text{op}}}: \mathbf{Mod}\text{-}\mathcal{C} \rightarrow \mathbf{Ab}$.

We give a concrete model for derived colimits based on the tensor product of \mathcal{C} -modules. We ignore the dual situation of derived limits completely, leaving it to the reader. For $L \in \mathbf{Mod}\text{-}\mathcal{C}$ and $M \in \mathcal{C}\text{-Mod}$, we define the *tensor product* $L \otimes_{\mathcal{C}} M$ to be the coequalizer of the action of \mathcal{C} on the two factors of the tensor product,

$$\bigoplus_{c_0, c_1 \in \mathcal{C}} Lc_1 \otimes \mathbb{Z}\mathcal{C}(c_0, c_1) \otimes Mc_0 \rightrightarrows \bigoplus_{c \in \mathcal{C}} Lc \otimes Mc$$

with the two maps sending $x \otimes f \otimes y$ to $xf \otimes y$ and to $x \otimes fy$.

In this way, the colimit functor can be seen as a tensor product $\text{colim}^{\mathcal{C}} M \cong \underline{\mathbb{Z}} \otimes_{\mathcal{C}} M$, where $\underline{\mathbb{Z}}$ denotes the constant diagram with the value \mathbb{Z} . By standard arguments, the derived colimit is defined by replacing either of $\underline{\mathbb{Z}}$ and M by a projective resolution. There is a canonical choice of a resolution of $\underline{\mathbb{Z}}$, given by $\mathbb{Z}N(-/\mathcal{C})$, the chain complex associated with the diagram of the nerves of the under categories c/\mathcal{C} . We will thus use as a definition

$$\text{colim}_n^{\mathcal{C}} M = H_n(\mathbb{Z}N(-/\mathcal{C}) \otimes_{\mathcal{C}} M).$$

Now, we discuss the dependence of the $\text{colim}_n^{\mathcal{C}}$'s on the indexing category. For each functor $F: \mathcal{D} \rightarrow \mathcal{C}$, there is an induced restriction functor $F^*: \mathbf{Mod}\text{-}\mathcal{C} \rightarrow \mathbf{Mod}\text{-}\mathcal{D}$; it admits both left and right adjoints, the Kan extension functors. Concretely, the left adjoint $F_!: \mathbf{Mod}\text{-}\mathcal{D} \rightarrow \mathbf{Mod}\text{-}\mathcal{C}$ is given by

$$F_!L(c) = L \otimes_{\mathcal{D}} \mathbb{Z}\mathcal{C}(c, F-).$$

Lemma 2. *The following formula holds*

$$F_!L \otimes_{\mathcal{C}} M \cong L \otimes_{\mathcal{D}} F^*M.$$

Proof. By the associativity of the tensor product, we have

$$\begin{aligned} F_!L \otimes_{\mathcal{C}} M &\cong (L \otimes_{\mathcal{D}} \mathbb{Z}\mathcal{C}(-, F-)) \otimes_{\mathcal{C}} M \\ &\cong L \otimes_{\mathcal{D}} (\mathbb{Z}\mathcal{C}(-, F-) \otimes_{\mathcal{C}} M) \cong L \otimes_{\mathcal{D}} MF = L \otimes_{\mathcal{D}} F^*M. \end{aligned} \quad \square$$

We will now make use of the above lemma to induce maps on derived colimits. The canonical map $\mathbb{Z}N(-/\mathcal{D}) \rightarrow F^*\mathbb{Z}N(-/\mathcal{C})$, given by F , has an adjunct $F_!\mathbb{Z}N(-/\mathcal{D}) \rightarrow \mathbb{Z}N(-/\mathcal{C})$ and upon tensoring with M yields

$$\mathbb{Z}N(-/\mathcal{D}) \otimes_{\mathcal{C}} F^*M \cong F_!\mathbb{Z}N(-/\mathcal{D}) \otimes_{\mathcal{C}} M \rightarrow \mathbb{Z}N(-/\mathcal{C}) \otimes_{\mathcal{C}} M$$

which induces in homology the map $F_*: \text{colim}_n^{\mathcal{D}} F^*M \rightarrow \text{colim}_n^{\mathcal{C}} M$.

Later, we will need the following simple observations.

Lemma 3. *There is an isomorphism*

$$\text{colim}_n^{\mathcal{D}} F^*M \cong H_n(\mathbb{Z}N(-/F) \otimes_{\mathcal{C}} M),$$

where c/F is the under category of F . In addition, $\mathbb{Z}N(-/F)$ is a chain complex of projectives.

Proof. It is simple to verify that $\mathbb{Z}N(-/F) \cong F_!\mathbb{Z}N(-/\mathcal{D})$. The first claim thus follows from Lemma 2. Since $F_!$ is left adjoint to an exact functor F^* , it preserves projectives, proving the second claim. \square

3. A counter-example

In this section, we present a counter-example to the main theorem of [1]. Thus, let (Cat) be a pushout square in which both functors F_1, F_2 are injective. In our example, we take $M = \mathbb{Z}$, the constant functor with value \mathbb{Z} . In this case, we have $\text{colim}_n^{\mathcal{C}} \mathbb{Z} \cong H_n(N\mathcal{C}; \mathbb{Z})$, the singular homology groups of the nerve of \mathcal{C} .

Let \mathcal{C}_0 be an arbitrary connected category for which the reduced homology $\tilde{H}_* N\mathcal{C}_0$ is non-trivial, e.g. \mathcal{C}_0 could be the group of integers. Then define \mathcal{C}_1 to be the category obtained from \mathcal{C}_0 by adjoining a disjoint initial object 0, while \mathcal{C}_2 is obtained by adjoining a disjoint terminal object 1. Then it is easy to see that the pushout \mathcal{C} contains both \mathcal{C}_1 and \mathcal{C}_2 as full subcategories and there is a unique morphism $0 \rightarrow 1$, making 0 an initial object and 1 a terminal object.

We consider the image of (Cat) under the nerve functor $N: \text{Cat} \rightarrow \mathbf{sSet}$. By the above considerations, $N\mathcal{C}_1, N\mathcal{C}_2$ and $N\mathcal{C}$ are all contractible, having thus zero reduced (co)homology. The exactness of any kind of Mayer–Vietoris sequence contradicts the triviality of these groups and non-triviality of $\tilde{H}_* N\mathcal{C}_0$.

4. Pushouts of categories

We reformulate the question of the existence of a Mayer–Vietoris sequence in terms of categories and spaces. We consider a square of categories (Cat) as above and, for any object $c \in \mathcal{C}$, form the square of nerves of under categories

$$\begin{array}{ccc} N(c/I_0) & \xrightarrow{(F_1)_*} & N(c/I_1) \\ (F_2)_* \downarrow & & \downarrow (I_1)_* \\ N(c/I_2) & \xrightarrow{(I_2)_*} & N(c/\mathcal{C}) \end{array} \quad (\mathbf{sSet}\text{-}\mathcal{C})$$

where $I_0: \mathcal{C}_0 \rightarrow \mathcal{C}$ is the composite functor in the original diagram, $I_0 = I_1 F_1 = I_2 F_2$. The significance of this diagram for the original problem lies in the following proposition, where we remind that a square is called *homotopy cocartesian* or a *homotopy pushout square* if the induced map from the homotopy pushout to the bottom right corner is a weak equivalence.

Proposition 4. *If the square $(\mathbf{sSet}\text{-}\mathcal{C})$ is homotopy cocartesian then (Cat) induces a Mayer–Vietoris sequence of derived (co)limits.*

Proof. We will work out the case of derived colimits, the case of limits is dual. Consider the category \mathbf{Ch} of non-negatively graded chain complexes with the projective model structure. The chain complex functor $\mathbb{Z}: \mathbf{sSet} \rightarrow \mathbf{Ch}$ is known to preserve homotopy colimits (the corresponding functor $\mathbb{Z}: \mathbf{sSet} \rightarrow \mathbf{sAb}$ is left Quillen and the Moore complex functor $\mathbf{sAb} \rightarrow \mathbf{Ch}$ is naturally quasi-isomorphic to the normalized chain complex functor; the latter is a Quillen equivalence) and therefore, the induced square of chain complexes

$$\begin{array}{ccc} \mathbb{Z}N(c/I_0) & \xrightarrow{(F_1)_*} & \mathbb{Z}N(c/I_1) \\ (F_2)_* \downarrow & & \downarrow (I_1)_* \\ \mathbb{Z}N(c/I_2) & \xrightarrow{(I_2)_*} & \mathbb{Z}N(c/\mathcal{C}) \end{array} \quad (\mathbf{Ch}\text{-}\mathcal{C})$$

is also homotopy cocartesian. By Lemma 3, all \mathcal{C} -modules appearing in $(\mathbf{Ch}\text{-}\mathcal{C})$ are projective. Tensoring with M over \mathcal{C} thus yields another homotopy pushout square of chain complexes. The homology groups of $\mathbb{Z}N(-/\mathcal{C}) \otimes_{\mathcal{C}} M$ are the derived colimits $\text{colim}_*^{\mathcal{C}} M$ by definition. For each $k = 0, 1, 2$, the homology groups of the chain complex $\mathbb{Z}N(-/I_k) \otimes_{\mathcal{C}} M$ are the derived colimits $\text{colim}_*^{\mathcal{C}_k}(I_k^* M)$ by Lemma 3. Thus, the Mayer–Vietoris sequence associated with the homotopy pushout square $(\mathbf{Ch}\text{-}\mathcal{C}) \otimes_{\mathcal{C}} M$ is the required sequence. \square

We will need a small lemma regarding the passage from (\mathbf{Cat}) to the induced square of under categories:

$$\begin{array}{ccc} c/I_0 & \xrightarrow{(F_1)_*} & c/I_1 \\ (F_2)_* \downarrow & & \downarrow (I_1)_* \\ c/I_2 & \xrightarrow{(I_2)_*} & c/\mathcal{C} \end{array} \quad (\mathbf{Cat}\text{-}\mathcal{C})$$

Lemma 5. *If (\mathbf{Cat}) is a pushout square, then so is $(\mathbf{Cat}\text{-}\mathcal{C})$.*

Proof. The functor $\mathbf{Cat}/\mathcal{C} \rightarrow \mathbf{Cat}$, $(\mathcal{D} \xrightarrow{F} \mathcal{C}) \mapsto c/F$, preserves colimits since it is a left adjoint: Let $Y_c: \mathcal{C} \rightarrow \mathbf{Set} \subseteq \mathbf{Cat}$ denote the representable functor $\mathcal{C}(c, -)$, thought of as a functor to \mathbf{Cat} by viewing each set as a discrete category. The value of the right adjoint $\mathbf{Cat} \rightarrow \mathbf{Cat}/\mathcal{C}$ on a category \mathcal{X} is given by the “lax” under category $Y_c//\mathcal{X}$, whose objects are pairs (a, φ) with $a \in \mathcal{C}$ and $\varphi: Y_c(a) \rightarrow \mathcal{X}$ and whose morphisms are pairs (f, τ) with $f: a \rightarrow b$ and $\tau: \varphi \Rightarrow \psi f_*$ a transformation. It is thought of as an object of \mathbf{Cat}/\mathcal{C} via $Y_c//\mathcal{X} \rightarrow \mathcal{C}$, $(a, \varphi) \mapsto a$. \square

Remark. An analogous lemma holds for the Grothendieck construction in \mathbf{Cat} : the under category of a Grothendieck construction is a Grothendieck construction of the under categories. Since the nerve functor takes, up to weak homotopy equivalence, Grothendieck constructions to homotopy colimits by [6], one obtains a Mayer–Vietoris sequence for derived colimits over a Grothendieck construction of a diagram $\mathcal{C}_1 \leftarrow \mathcal{C}_0 \rightarrow \mathcal{C}_2$.

In fact, it is possible to extend the above to more general shapes than just spans of categories, at the cost of replacing the Mayer–Vietoris sequence by a spectral sequence for homology groups of a homotopy colimit of chain complexes. Denoting the Grothendieck construction by $\mathcal{C} = \int^{i \in \mathcal{I}} \mathcal{C}_i$, this becomes

$$\mathrm{colim}_p^{\mathcal{I}}(i \mapsto \mathrm{colim}_q^{\mathcal{C}_i} M) \Longrightarrow \mathrm{colim}_{p+q}^{\mathcal{C}} M.$$

Such a spectral sequence was obtained in a more general context in [4].

However, the main point of our theorem is to simplify the computation of derived colimits over a *given* category \mathcal{C} by splitting it into subcategories, a procedure rather inverse to the above.

5. Local coverings of categories

There is an obvious candidate for a condition on the square (\mathbf{Cat}) that ensures that $(\mathbf{sSet}\text{-}\mathcal{C})$ is homotopy cocartesian. Namely, by a theorem of Thomason [5] on the Quillen equivalence $\mathbf{Cat} \simeq_Q \mathbf{sSet}$, we may translate this question to \mathbf{Cat} with Thomason’s model structure. In a left proper model category such as \mathbf{Cat} , a pushout square is homotopy cocartesian if one of the functors F_1, F_2 is a cofibration. This, however, is too restrictive, since cofibrations in \mathbf{Cat} are rather rare. In particular, this does not apply to the case of a pushout square of group monomorphisms that we would like to generalize.

The main property of group monomorphisms used in the proof of [2, 1B.12] is that on nerves, they give coverings. We will now generalize this notion from groups to arbitrary categories.

Definition 6. A functor $F: \mathcal{D} \rightarrow \mathcal{C}$ is said to be a *local covering* if each under category c/F is homotopically discrete, i.e. weakly homotopy equivalent to a discrete space.

Example 7. Let $F: \mathcal{D} \rightarrow \mathcal{C}$ be a faithful functor from a groupoid \mathcal{D} into a category \mathcal{C} in which every morphism is epi. Then F is a local covering, as follows from an easily verified fact that each c/F is a groupoid in which every automorphism is the identity.

In particular, any faithful functor between groupoids is a local covering.

Remark. The condition from the definition is named a local covering since it does not give any (global) information on $F_*: N\mathcal{D} \rightarrow N\mathcal{C}$. We say that F is a *covering* if, in addition to the local covering condition, for every arrow $c' \rightarrow c$, the induced functor $c'/F \rightarrow c/F$ is bijective on π_0 , i.e.

$N(c/F) \rightarrow N(c'/F)$ is a weak homotopy equivalence. Quillen's Theorem B then shows that the homotopy fibre of $F_*: N\mathcal{D} \rightarrow N\mathcal{C}$ is also homotopically discrete and thus, F_* is a covering.

If F_k in (Cat) is a covering then Quillen's Theorem B also yields a fibration sequence

$$N(c_k/F_k) \rightarrow N(c/I_0) \rightarrow N(c/I_k);$$

when I_k is a (local) covering, both the base and the fibre are homotopy discrete, hence also the total space, and I_0 is a (local) covering too. We believe that if both F_1, F_2 are coverings then so are I_1, I_2 and, by the above, also I_0 .

We remind that a *homotopy 1-type* is a Kan complex K with $\pi_n K = 0$ for all $n > 1$. It is possible to associate to every simplicial set X a homotopy 1-type $P_1 X$ (the first Postnikov section of X) by killing all higher homotopy groups of X . For another point of view on this construction, see Proposition 9. The following lemma contains the heart of the proof of Theorem 1.

Lemma 8. *For any pushout square of categories (Cat) with at least one of F_1, F_2 injective on objects, the map from the homotopy pushout in $(\text{sSet-}\mathcal{C})$ to its bottom right corner $N(c/\mathcal{C})$ is a 2-equivalence, i.e. the associated map of homotopy 1-types is a homotopy equivalence.*

Proof. By Proposition 9, there is a Quillen equivalence

$$\tau_1: \text{sSet}_{\text{loc}} \rightleftarrows \text{Cat}_{\text{loc}} : N.$$

The condition from the statement is equivalent to the square $(\text{sSet-}\mathcal{C})$ being homotopy cocartesian in sSet_{loc} . This happens if and only if its image under τ_1 is homotopy cocartesian. In the localized model structure, $\tau_1 P_1 N\mathcal{D} \simeq \tau_1 N\mathcal{D} \cong \mathcal{D}$. Thus, we are left to verify that $(\text{Cat-}\mathcal{C})$ is homotopy cocartesian in Cat_{loc} and for that matter, we localize this square. This produces a pushout square since the localization $\mathcal{D} \mapsto \mathcal{D}[\mathcal{D}^{-1}]$ commutes with colimits (it is a reflection of categories into groupoids). It is known that a pushout square is homotopy cocartesian if one of the maps $(F_1)_*, (F_2)_*$ is a cofibration. In Cat_{loc} , these are exactly functors that are injective on objects and it is trivial to check that this condition for F_k implies that for $(F_k)_*: c/I_0 \rightarrow c/I_k$. \square

Proof of Theorem 1. By definition of local coverings, all spaces appearing in the square $(\text{sSet-}\mathcal{C})$ are homotopy discrete. Therefore, up to homotopy, its homotopy pushout P is 1-dimensional and thus a homotopy 1-type. The contractible $N(c/\mathcal{C})$ is also a homotopy 1-type and the previous lemma gives $P \simeq N(c/\mathcal{C})$. Hence, $(\text{sSet-}\mathcal{C})$ is homotopy cocartesian and by Proposition 4, it induces a Mayer–Vietoris sequence. \square

A. Homotopy 1-types and groupoids

We were not able to find the following statements anywhere explicitly.

Proposition 9. *There exist left Bousfield localizations of the categorical model structure on Cat and of the Quillen model structure on sSet such that*

- *local objects in Cat are exactly the groupoids and the localization is $\mathcal{D} \rightarrow \mathcal{D}[\mathcal{D}^{-1}]$,*
- *local objects in sSet are exactly homotopy 1-types and the localization is $X \rightarrow P_1 X$,*
- *there is a Quillen equivalence $\tau_1: \text{sSet}_{\text{loc}} \rightleftarrows \text{Cat}_{\text{loc}} : N$.*

Proof. Using [3], the categorical model structure on Cat is left Bousfield localized with respect to $\tau_1 \Lambda_0^2 \rightarrow \tau_1 \Delta^2$ and $\tau_1 \Lambda_2^2 \rightarrow \tau_1 \Delta^2$, the images under τ_1 of the two outer horn inclusions. Similarly, the Quillen model structure is localized with respect to $\partial \Delta^n \rightarrow \Delta^n$ for all $n > 2$.

The first two points are easy to verify. For the last point, we observe that the localization of $\tau_1 X$ is the fundamental groupoid $\Pi_1 X$, i.e. $\tau_1 \simeq \Pi_1$, and thus, τ_1 preserves and reflects local equivalences. Consequently, it is enough to verify that the derived counit is a weak equivalence. The counit $\tau_1 N \rightarrow \text{Id}$ is known to be an isomorphism and on local objects, it is weakly equivalent to the derived counit since all simplicial sets are cofibrant. \square

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